Lower bounds for the ground-state degeneracies of frustrated systems on fractal lattices

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(Received 28 July 2000)

The total number of ground states for nearest-neighbor-interaction Ising systems with frustrations, defined on hierarchical lattices, is investigated. A simple method is presented, which allows one to factorize the ground-state degeneracy, at a given hierarchy level n, in terms of contributions due to all hierarchy levels. Such a method may yield the exact ground-state degeneracy of uniformly frustrated systems, whereas it works as an approximation for randomly frustrated models. In the latter cases, it is demonstrated that such an approximation yields lower-bound estimates for the ground-state degeneracies.

PACS number(s): 64.60.Ak, 75.10.Nr, 05.50.+q, 65.50.+m

Magnetic models presenting frustration have called the attention of many workers during the last decades [1]. The frustration concept [2] results from the competition between interactions, leading to the possibility of several minimum-energy configurations. Many real systems display frustration, like ice [1], spin glasses [3–5], and diluted antiferromagnets [6]. In spin glasses, frustration effects are combined with disorder, leading to a highly nontrivial low-temperature phase, with many metastable states, usually associated with a very slow dynamics.

Many frustrated systems present an extensive ground-state (g.s.) degeneracy, in such a way that the total number of g.s.'s increases exponentially with the number of sites N. For disordered models, the average number of g.s.'s, $[N_{\mathrm{g.s.}}]_{av}$ (herein $[\ \ ]_{av}$ represents an average over an arbitrary type of disorder: for random magnetic fields, one has an average over the field probability distribution, whereas in the case of spin glasses, one has an average over the coupling probability distribution), behaves like

$$[N_{\mathrm{g.s.}}]_{av} \sim \exp(hN), \tag{1}$$

where h is some positive finite number [in the case of Ising systems, $0 \leqslant h \leqslant \ln 2$, since the maximum number of states is $2^N = \exp(N \ln 2)$]. For uniform systems (with no randomness), there is no average over the disorder, in such a way that, in the thermodynamic limit, one gets that $h = s_0$, where s_0 denotes the zero-temperature entropy per particle, usually denoted the residual entropy (herein we work in units of $k_B = 1$). For randomly frustrated systems, due to the averaging process, h is not related to the g.s. entropy, but rather to the g.s. complexity [7]. A behavior similar to the one of Eq. (1) holds for the infinite-range-interaction Ising spin glass [8], with $h \approx 0.20$ [9]. The applicability of Eq. (1) for nearest-neighbor-interaction Ising spin glasses on Bravais lattices represents, at the moment, a controversial matter [5].

The average number of g.s.'s has been estimated for short-range Ising spin glasses on diamond hierarchical lattices (HL's), with different probability distributions for the couplings [10]: one finds a zero g.s. complexity per particle in the case of continuous probability distributions, whereas for a bimodal $(\pm J)$ distribution, an exponential increase in $[N_{\mathrm{g.s.}}]_{av}$ has been verified, on lattices of fractal dimensions $d_l \leqslant d \leqslant 5$ (where $d_l \approx 2.58$ represents the respective spin-glass lower critical dimension), with h varying roughly from 0.16 (for $d = d_l$) to 0.27 (for $d = 5$). Obviously, the exponential increase of Eq. (1) is expected to hold for diamond HL's with any fractal dimension $d \geqslant d_l$. Such a calculation was performed within the now called Factorization Approach (FA), for which the total number of g.s.'s at hierarchy level n is expressed as a product of properly defined partial number of g.s.'s at hierarchy levels $n, n-1, \ldots, 1$. The FA is in general an approximation, and it leads to the exact g.s. degeneracy only for very simple systems [11]. Another method, denoted herein as the recursive approach (RA) [11], allows one to calculate the g.s. degeneracy through exact recursion relations, based on the recursive properties of the particular HL. Recently, the g.s. entropy of several uniformly frustrated Ising antiferromagnets on fractal lattices has been calculated exactly through the RA [11]. It was shown that, for some simple models, the FA yields the correct residual entropy, whereas in other cases, it leads to a lower estimate, as compared with the one obtained through the RA. Herein, we discuss under which conditions the FA yields the correct estimate for the g.s. degeneracy of Ising systems on arbitrary HL's; in particular, we demonstrate that, in the case of randomly frustrated systems, the FA estimate represents always a lower bound.

Let us now introduce the two above-mentioned methods for calculating the g.s. degeneracies of frustrated systems. The RA is based on the recursive properties of the particular HL; the central idea is to express g.s. degeneracies at a given hierarchy level in terms of those of the previous hierarchy. For a given disorder configuration, one may fix the spins of the zeroth hierarchy level (at this level, the number of terminal spins of a given unit cell of the HL is defined), in such a

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way so as to obtain a set of $\{G_\alpha^{(k)}\}$ possible degeneracies at an arbitrary hierarchy level k; for each value of α one has a recursion relation

$$G_\alpha^{(k)}=\Psi_\alpha(G_1^{(k-1)},G_2^{(k-1)},\ldots). \tag{2}$$

Since one may compute easily the set of degeneracies at hierarchy level 1 (at this level, the basic unit cell of the HL is generated), $\{G_\alpha^{(1)}\}\equiv\{g_\alpha\}$, the recursion relations in Eq. (2) may be followed up to any desired hierarchy level. For a fixed disorder configuration, the total number of g.s.'s of the HL at its nth hierarchy level is expressed as

$$N_{\text{g.s.}}^{(n)}=\sum_\alpha a_\alpha G_\alpha^{(n)}, \tag{3}$$

where the coefficients a_α count how many different configurations of the spins at the zeroth level contribute to the same $G_\alpha^{(n)}$. Finally, if one is dealing with a random system, an average over the disorder has to be performed, in such a way so as to obtain $[N_{\text{g.s.}}^{(n)}]_{av}$.

If one succeeds in obtaining the recursion relations in Eq. (2) exactly, the RA yields the exact number of g.s.'s of the HL at its nth hierarchy level, as defined by Eq. (3). However, this turns out to be a difficult task in some cases, like for random systems, where one has to deal with the disorder average. In such cases, the FA appears as a very convenient technique.

Herein, we shall introduce the FA for the general case of an arbitrary randomly frustrated system. Before doing that, we shall present a few definitions. Let us consider a given HL at its nth hierarchy level; one may count partially the number of g.s.'s of the HL by fixing the terminal spins of each unit cell, in such a way that they correspond to one specific g.s. configuration μ of the HL at hierarchy level $n-1$. One may easily see that, for an arbitrary level k, the number of g.s. configurations obtained by fixing the spins of level $k-1$ (i.e., the increase of g.s. configurations in going from hierarchy $k-1$ to hierarchy k) may be written as

$$\Gamma^{(k)}(\mu)=\prod_\alpha (g_\alpha)^{N_{c,\alpha}^{(k)}(\mu)}, \tag{4}$$

where $N_{c,\alpha}^{(k)}(\mu)$ denotes the number of unit cells, at the particular g.s. configuration μ of level $k-1$, with degeneracy g_α, in the HL at its kth hierarchy level. It is important to recall that, whenever applicable, the trivial case $g_\alpha=1$ (nonfrustrated cell) is included in Eq. (4) in such a way that for each configuration μ, $\Sigma_\alpha N_{c,\alpha}^{(k)}(\mu)=N_c^{(k)}$, where $N_c^{(k)}$ represents the total number of unit cells of the HL at hierarchy level k. The average over a set of g.s. configurations $\{\mu\}$ (herein denoted by $\langle\rangle$), of hierarchy level $k-1$, becomes

$$\langle\Gamma^{(k)}\rangle=\frac{1}{N_{\text{g.s.}}^{(k-1)}}\sum_{\mu=1}^{N_{\text{g.s.}}^{(k-1)}}\Gamma^{(k)}(\mu). \tag{5}$$

On the other hand, from the definition of $\Gamma^{(k)}(\mu)$ in Eq. (4), one sees that the total number of g.s.'s of the HL at its nth level is given by

$$N_{\text{g.s.}}^{(n)}=\sum_{\mu=1}^{N_{\text{g.s.}}^{(n-1)}}\Gamma^{(n)}(\mu)=\sum_{\mu=1}^{N_{\text{g.s.}}^{(n-1)}}\prod_\alpha (g_\alpha)^{N_{c,\alpha}^{(n)}(\mu)}. \tag{6}$$

One may easily see that, if one succeeds in obtaining the recursion relations of Eq. (2), within the RA, the total number of g.s.'s of Eq. (3) may be also written in the form of the equation above. Comparing Eqs. (5) and (6), one gets the recursion relation

$$N_{\text{g.s.}}^{(n)}=N_{\text{g.s.}}^{(n-1)}\langle\Gamma^{(n)}\rangle, \tag{7}$$

which may be iterated to yield

$$N_{\text{g.s.}}^{(n)}=\langle\Gamma^{(n)}\rangle\langle\Gamma^{(n-1)}\rangle\langle\Gamma^{(n-2)}\rangle\cdots\langle\Gamma^{(1)}\rangle A, \tag{8}$$

where the factor A corresponds to the number of states associated to hierarchy level 0. The FA is represented in the equation above, expressing the total number of g.s.'s of a HL at its nth hierarchy level as a factorization of contributions due to all hierarchy levels $(k=n,n-1,\ldots,1,0)$. It should be mentioned that, up to this point, both methods (RA and FA) yield the same (exact) g.s. degeneracy estimate.

Now, let us substitute Eq. (4) into Eq. (5); one gets that

$$\begin{aligned}\langle\Gamma^{(k)}\rangle&=\frac{1}{N_{\text{g.s.}}^{(k-1)}}\sum_{\mu=1}^{N_{\text{g.s.}}^{(k-1)}}\exp\left(\sum_\alpha N_{c,\alpha}^{(k)}(\mu)\ln g_\alpha\right)\\&\geqslant\exp\left(\sum_\alpha \phi_\alpha^{(k)}\ln g_\alpha\right)=\prod_\alpha (g_\alpha)^{\phi_\alpha^{(k)}},\end{aligned} \tag{9}$$

where the general property of the exponential function, $\langle\exp(x)\rangle\geqslant\exp(\langle x\rangle)$, has been used, and we have defined

$$\phi_\alpha^{(k)}=\langle N_{c,\alpha}^{(k)}\rangle=\frac{1}{N_{\text{g.s.}}^{(k-1)}}\sum_{\mu=1}^{N_{\text{g.s.}}^{(k-1)}}N_{c,\alpha}^{(k)}(\mu), \tag{10}$$

for the average, over g.s. configurations, of the number of cells with degeneracy g_α; obviously, one has that $\Sigma_\alpha\phi_\alpha^{(k)}=N_c^{(k)}$. In simple uniform models, $N_{c,\alpha}^{(k)}(\mu)$ may be calculated exactly for each g.s. configuration μ [11], whereas in more complicated uniform [11] or random problems [10], one may need to replace $N_{c,\alpha}^{(k)}(\mu)$ by $\phi_\alpha^{(k)}$. In the latter cases, the FA yields lower-bound estimates for the total number of g.s.'s, i.e.,

$$N_{\text{g.s.}}^{(n)}\geqslant A\prod_{k=1}^{n}\left(\prod_\alpha (g_\alpha)^{\phi_\alpha^{(k)}}\right). \tag{11}$$

Let us now consider a randomly frustrated system, in which case one needs to apply an average over the disorder; in the equation above one has that

$$[N^{(n)}_{\rm g.s.}]_{av} \geqslant A\left[\prod_{k=1}^{n}\prod_{\alpha}(g_\alpha)^{\phi_\alpha^{(k)}}\right]_{av}$$
$$=\left[\exp\left(\sum_{k=1}^{n}\sum_{\alpha}\phi_\alpha^{(k)}\ln g_\alpha\right)\right]_{av}$$
$$\geqslant \exp\left(\sum_{k=1}^{n}\sum_{\alpha}[\phi_\alpha^{(k)}]_{av}\ln g_\alpha\right), \qquad (12)$$

where we have used, similarly to what was done in Eq. (9), the convexity property of the exponential function. It is important to point out that the averages over the disorder in Eq. (12) refer to hierarchy level n (in $[N^{(n)}_{\rm g.s.}]_{av}$) and to each hierarchy level k (in $[\phi_\alpha^{(k)}]_{av}$), since the probability distribution associated with the disorder is renormalized at each hierarchy level.

Let us now define the FA for random systems [10] as an approximation which allows one to factorize the average number of ground states as

$$[N^{(n)}_{\rm g.s.FA}]_{av} = A\prod_{k=1}^{n}\left(\prod_{\alpha}(g_\alpha)^{[\phi_\alpha^{(k)}]_{av}}\right), \qquad (13)$$

which, according to Eq. (12), yields a lower-bound estimate for average number of g.s.'s of the HL,

$$[N^{(n)}_{\rm g.s.}]_{av} \geqslant [N^{(n)}_{\rm g.s.FA}]_{av}. \qquad (14)$$

It is important to recall that the above result holds for an arbitrary HL. This ensures that the results of Ref. [10], for nearest-neighbor-interaction $\pm J$ Ising spin glasses on diamond HL's, are indeed lower-bound estimates; the FA yields, for the $d=3$ diamond HL, the exponential increase of Eq. (1) with $h \approx 0.208$ [10]. The result $[N^{(n)}_{\rm g.s.}]_{av}=2$ if $d > 2.58$, for continuous probability distributions for the couplings [10], is in full agreement with recent investigations of the Ising spin glass, with a Gaussian probability distribution for the couplings, on the diamond HL of fractal dimension $d=3$, which find evidence of a simple spin-glass phase (characterized by two time-reversed states only) [12]. However, the results of Ref. [10], for the average number of ground states of $\pm J$ Ising spin glasses on diamond HL's, which are demonstrated herein to represent lower-bound estimates—implying that this average number increases effectively as an exponential of N—strongly suggest a different picture for the spin-glass phase of such systems.

PCI/MCT, CNPq, and Pronex/MCT (Brazilian agencies) are acknowledged for partial financial support. F.D.N. acknowledges the people at the CBPF (Centro Brasileiro de Pesquisas Físicas) for the warm hospitality received during a visit in which this work was developed.

[1] R. Liebmann, *Statistical Mechanics of Periodic Frustrated Ising Systems*, Lecture Notes in Physics Vol. 251 (Springer-Verlag, Berlin, 1986).

[2] G. Toulouse, Commun. Phys. **2**, 115 (1977); R. J. V. dos Santos and M. L. Lyra, Physica A **182**, 133 (1992).

[3] K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).

[4] K. H. Fischer and J. A. Hertz, *Spin Glasses* (Cambridge University Press, Cambridge, England, 1991).

[5] *Spin Glasses and Random Fields*, edited by A. P. Young (World Scientific, Singapore, 1998).

[6] D. P. Belanger, in *Spin Glasses and Random Fields*, edited by A. P. Young (World Scientific, Singapore, 1998).

[7] R. G. Palmer, Adv. Phys. **31**, 669 (1982).

[8] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).

[9] A. J. Bray and M. A. Moore, J. Phys. C **13**, L369 (1980); C. De Dominicis, M. Gabay, T. Garel, and H. Orland, J. Phys. (Paris) **41**, 923 (1980); F. Tanaka and S. F. Edwards, J. Phys. F: Met. Phys. **10**, 4630 (1977).

[10] E. M. F. Curado, F. D. Nobre, and S. Coutinho, Phys. Rev. E **60**, 3761 (1999).

[11] F. D. Nobre and E. M. F. Curado, Phys. Rev. E **62**, 4597 (2000).

[12] M. A. Moore, H. Bokil, and B. Drossel, Phys. Rev. Lett. **81**, 4252 (1998); E. Marinari, G. Parisi, J. J. Ruiz-Lorenzo, and F. Zuliani, *ibid.* **82**, 5176 (1999); H. Bokil, A. J. Bray, B. Drossel, and M. A. Moore, *ibid.* **82**, 5177 (1999); B. Drossel, H. Bokil, M. A. Moore, and A. J. Bray, Eur. Phys. J. B **13**, 369 (2000).